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THE STEADY VIBRATIONS AND RESISTANCE OF A RAILWAY TRACK TO THE UNIFORM MOTION OF AN UNBALANCED WHEEL†

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A model of a railway track, in the form of an infinite Timoshenko beam resting on equally spaced massive visco-elastic supports, is considered. Steady vertical vibrations of the track due to a harmonic force moving along it at a constant velocity are investigated. The vertical displacement of the track is represented in a moving system of coordinates by a generalized Fourier series. The steady vertical vibrations of a massive rigid wheel rolling along the track at a constant velocity and loaded by a vertical harmonic force are investigated. The track-wheel interaction force is expressed as a generalized Fourier series whose coefficients are determined using an equality relating the vertical displacements of the wheel and the track. Vibrations of the wheel due to centrifugal force and periodic changes in the track parameters are considered. Parametric vibrations of a wheel moving at a constant velocity under a static load due to periodic variation in the stiffness of the track are investigated. The force with which the track resists the uniform motion of an unbalanced wheel is computed.

1. THE TIMOSHENKO BEAM

Timoshenko's theory [1, 2] makes allowance for both bending and shear deformations of a beam whose elastic axis experiences discontinuities in the first and third derivatives at the points of application of concentrated transverse forces. This theory was reduced in [1] to a system of two partial differential equations and applied to the problem of the free vibrations of a single-span beam; there were no concentrated transverse forces or discontinuities in the derivatives. The static displacement of a beam loaded by concentrated transverse forces was computed in [2]; the problem was reduced to a single ordinary differential equation, which simplified allowance for discontinuities.

We shall reduce this equation to a more general form by adding terms designed to take into account both a transverse load and moments, either concentrated or distributed along the beam. An element of the beam, loaded by a distributed moment $m = m(x)$ and distributed transverse force $q = q(x)$, bounded by cross-sections at x and $x + dx$, is shown in Fig. 1. The figure also shows the positive directions of the bending moment M, of shearing force Q in a cross-section of the beam, and also of the angles of rotation φ and dy/dx of the cross-section of the beam and its elastic axis $y = y(x)$.

Under the action of the shearing force Q, the rectangular beam element becomes a parallelogram, and adjacent sides rotate relative to one another (Fig. 1) through an angle $\gamma = Q/R$. The quantity $R = k'GA$ is known as the shear stiffness, where G is the shear modulus, A is the area of cross-section of the beam, and the coefficient k' allows for non-uniform distribution of the shearing force over the cross-section. The quantities φ, γ and dy/dx satisfy the equality

$$dy/dx = \varphi - \gamma = \varphi - Q/R \tag{1.1}$$

At the point where the concentrated transverse force is applied to the beam, the derivative dy/dx is undefined, while the shearing force Q experiences a jump. By the last equality, the angle of rotation dy/dx of the elastic axis of the beam also experiences a jump at that point. These jumps are related by the equality

$$[Q] = -R[dy/dx] \tag{1.2}$$

(the square brackets denote the jump in the bracketed quantity, that is, the difference between its right and left limits). It follows from the equilibrium conditions $qdx - dQ = 0$ and $mdx + dM - Qdx = 0$ of the beam element that

$$q = dQ/dx, \quad Q = dM/dx + m, \quad d^2M/dx^2 = dQ/dx - dm/dx = q(x) - dm/dx$$

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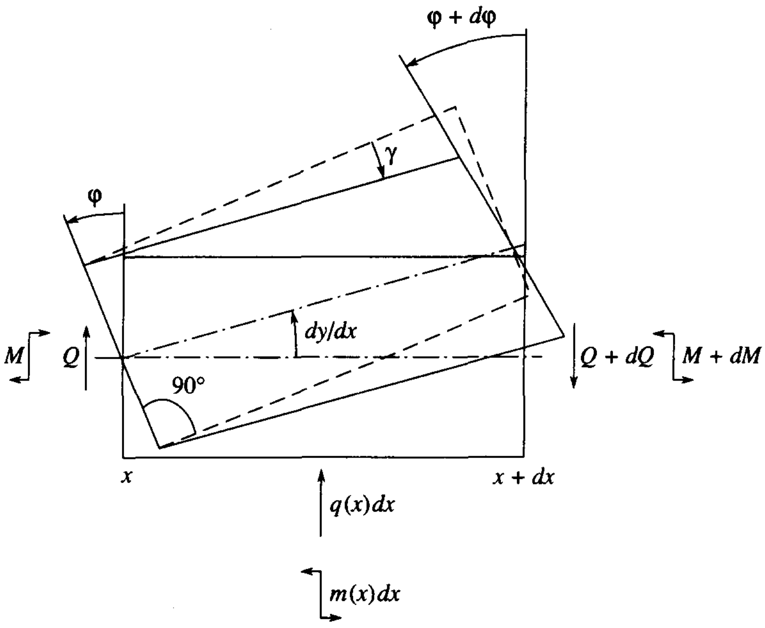

Fig. 1

The bending displacement of the beam is defined by the equality

$$M = EJd\varphi/dx \tag{1.3}$$

where EJ is the bending stiffness of the beam, E is its modulus of elasticity, and J is the moment of inertia of the beam cross-section. Differentiating Eq. (1.1) twice and using the equilibrium conditions of the beam element and Eq. (1.3), we obtain

$$d^3y/dx^3 = (EJ)^{-1}dM/dx - R^{-1}dq/dx = (EJ)^{-1}(Q - m) - R^{-1}dq/dx$$

whence it follows that at the point where the concentrated transverse force is applied to the beam,

$$[Q] = EJ[d^3y/dx^3] \tag{1.4}$$

A third differentiation yields the equation

$$d^4y/dx^4 - (EJ)^{-1}(q - dm/dx) + R^{-1}d^2q/dx^2 = 0 \tag{1.5}$$

which determines the transverse displacement of the beam.

2. THE STEADY VIBRATIONS OF A RAILWAY TRACK

We will represent the track as a Timoshenko beam with a density per unit length ρ_0. Let t and $y(x, t)$ denote the time and transverse displacement of the beam, respectively. We replace the quantity $q = q(x)$ in Eq. (1.5) by the expression $q(x, t) - \rho_0\partial^2y(x, t)/\partial t^2$, in which the first term is the external transverse load and the second is the inertia force of the transverse motion of the beam. Let $m = m(x)$ denote the moment $-(\rho_0 J/A)\partial^2\varphi/\partial t^2$ of the forces of inertia of rotation of the beam cross-section. Finally, we obtain an equation for the vertical vibrations of the track:

$$EJ\frac{\partial^4 y(x,t)}{\partial x^4} + \rho_0\frac{\partial^2 y(x,t)}{\partial t^2} - \rho_0\left(\frac{J}{A} + \frac{EJ}{R}\right)\frac{\partial^4 y(x,t)}{\partial x^2\partial t^2} + \frac{\rho_0^2 J}{RA}\frac{\partial^4 y(x,t)}{\partial t^4} = q(x,t) + \frac{\rho_0 J}{RA}\frac{\partial^2 q(x,t)}{\partial t^2} - \frac{EJ}{R}\frac{\partial^2 q(x,t)}{\partial x^2} \tag{2.1}$$

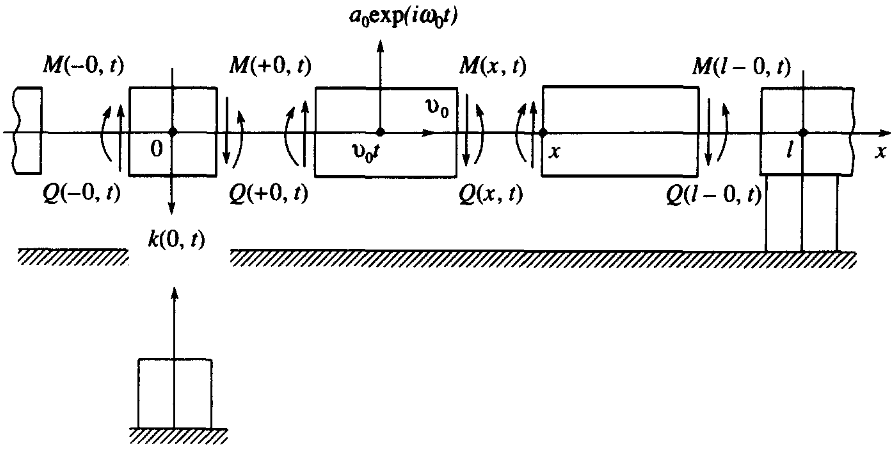

Fig. 2

Equation (2.1) was obtained previously in [3] by introducing two independent variables y and φ and subsequently eliminating φ. The application of Eq. (1.5), as well as equalities (1.2) and (1.4), leads directly to Eq. (2.1) and enables us to formulate a boundary-value problem for this equation.

Suppose the track is resting on sleepers with spacing l and is disturbed by a vertical concentrated harmonic force $a_0\exp(i\omega_0 t)$ moving along the track without detachment at a constant velocity υ_0. If $x = 0$ corresponds to a sleeper and $t = 0$ corresponds to the time at which the force passes over that sleeper (Fig. 2), the transverse load on the track may be expressed as

$$q(x, t) = a_0\exp(i\omega_0 t)\delta(x - \upsilon_0 t)$$

The Dirac function $\delta(x - \upsilon_0 t)$ defines the concentrated disturbing force at the point $x_0 = \upsilon_0 t$. The quantity $\Phi_0 = \omega_0 l/\upsilon_0$ equals the phase increment of the harmonic force in the time l/υ_0 that the force takes to traverse the distance l. During that time the harmonic force acquires the factor $\exp(i\Phi_0)$. The steady vertical displacement $y(x, t)$ of the track acquires the same factor [4, 5]. Therefore,

$$y(x + l, t + l/\upsilon_0) = \exp(i\Phi_0)y(x, t) \tag{2.2}$$

Relation (2.2) defines a solution of the equation of vertical vibrations of the track (2.1) which is bounded as $t \to \pm\infty$. If $\omega_0 = 2\pi p_1\upsilon_0/l$, where p_1 is an integer, then the frequency of the disturbing force is a multiple of the sleeper passing frequency υ_0/l. In that case $\exp(i\Phi_0) = 1$, and condition (2.2) becomes the condition for periodic vibrations [6]

$$y(x + l, t + l/\upsilon_0) = y(x, t)$$

Steady vibrations of the track due to uniform movement of a constant transverse force lead to the same condition [7].

Suppose the frequency of the disturbing force and the sleeper passing frequency are commensurate, their quotient being equal to a rational number p_1/p_2, where p_1 and p_2 relatively prime integers. Repeated use of relation (2.2) yields the condition

$$y(x + p_2 l, t + p_2 l/\upsilon_0) = y(x, t) \tag{2.3}$$

which generalizes the condition for periodic vibrations and shows that the quantity $y(x, t)$ is transformed into itself under a single translation by $p_2 l$ with respect to x and $p_2 l/\upsilon_0$ with respect to t. If the frequency of the disturbing force and the sleeper passing frequency are not commensurate, the displacement of the track will be periodic neither in the fixed coordinate system nor in a coordinate system attached to the moving harmonic force.

We shall assume that the reaction of the support at $x = 0$ reduces to a vertical force $k(0, t)$ whose positive direction is shown in Fig. 2. At a high frequency of the disturbing force and high velocity of motion, sleeper bending must be taken into account, as well as wave propagation in the ballast and the ground [5, 8]. To that end, a two-mass model of track support is used [9]. At moderate values of these quantities, the sleeper may be represented by a single mass supported by a spring and a damper in parallel. Then

$$k(0, t) = \rho_1 l\partial^2 y(0, t)/\partial t^2 + rl\partial y(0, t)/\partial t + uly(0, t) \tag{2.4}$$

The quantities $\rho_1 l$, rl and ul are the support mass, damper viscosity and spring stiffness; they are expressed in terms of the parameters of a suitable uniform viscoelastic foundation.

At $x = 0$ the shearing force experiences a jump discontinuity of $-k(0, t)$. By Eqs (1.2) and (1.4), the functions $EJ\partial^3 y(x, t)/\partial x^3$ and $-R\partial y(x, t)/\partial x$ experience the same jump discontinuity at $x = 0$. The values of these functions to the left of the supports at points $x = 0$ and $x = l$ satisfy condition (2.2). Changing at $x = 0$ to the values of these functions to the right of the support, we obtain four boundary conditions

$$\partial^n y(l, t + l/v_0)/\partial x^n = \exp(i\Phi_0)(\partial^n y(0, t)/\partial x^n + \kappa_n k(0, t))$$

$$n = 0, 1, 2, 3; \quad \kappa_0 = 0,\ \kappa_1 = -R^{-1},\quad \kappa_2 = 0,\quad \kappa_3 = (EJ)^{-1}$$

We shall assume that each point of the beam was at rest before the approach of the disturbing force and, as a result of the viscous resistance in the supports, it returns to that state after the force has departed. Thus, $y(x, t) \to 0$ as $t \to \pm\infty$ together with its derivatives, so that the function admits of a Fourier transform with respect to t.

We substitute $q(x, t)$ into Eq. (2.1) and change to dimensionless variables and the parameters

$$X = x/l,\quad T = v_0 t/l,\quad Y(X, T) = y(x, t)/l,\quad K(0, T) = k(0, t)l^2/(EJ)$$

Note that the dimensionless coordinates $X_0 = x_0/l = v_0 t/l$ of the point of application of the disturbing force coincides with the dimensionless time T. We define the dimensionless amplitude of the disturbing force as $A_0 = a_0 l^2/(EJ)$ and note that

$$\delta(l(X - T)) = \delta(X - T)/l,\quad \delta(T - X) = \delta(X - T)$$

The Fourier transformation and the inverse Fourier transformation are defined as

$$\int_{-\infty}^{+\infty} Y(X, T)\exp(-i\Phi T)dT = Y^*(X),\quad Y(X, T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y^*(X)\exp(i\Phi T)d\Phi \tag{2.5}$$

where Φ is a dimensionless parameter. We then put

$$\int_{-\infty}^{+\infty} K(0, T)\exp(-i\Phi T)dT = K(\Phi)Y^*(0),\quad K(\Phi) = K_0 + iK_1\Phi - K_2\Phi^2$$

$$K_0 = ul^4/(EJ),\quad K_1 = rv_0 l^3/(EJ),\quad K_2 = \rho_1 v_0^2 l^2/(EJ)$$

The form of the right-hand side of the first equality follows from the linearity of the problem. Taking Fourier transformations in the equation of vertical vibrations of the track and in the four boundary conditions, we get

$$\begin{aligned} &d^4Y^*(X)/dX^4 + (\mathrm{B} + \Gamma)d^2Y^*(X)/dX^2 + (\mathrm{B}\Gamma - \mathrm{A})Y^*(X) = \\ &= A_0(1 + \psi((\Phi_0 - \Phi)^2 - \mathrm{B}))\exp(i(\Phi_0 - \Phi)X) \end{aligned} \tag{2.6}$$

$$\mathrm{A} = \rho_0 v_0^2 l^2 \Phi^2/(EJ),\quad \mathrm{B} = \rho_0 v_0^2 \Phi^2/(EA),\quad \Gamma = \rho_0 v_0^2 \Phi^2/R$$

$$d^n Y^*(1)/dX^n = \exp((i(\Phi_0 - \Phi))d^n Y^*(0)/dX^n + \mathrm{K}_n K(\Phi)Y^*(0)) \tag{2.7}$$

$$n = 0, 1, 2, 3;\quad \mathrm{K}_0 = 0,\quad \mathrm{K}_1 = -\psi,\quad \mathrm{K}_2 = 0,\quad \mathrm{K}_3 = 1,\quad \psi = \Gamma/\mathrm{A}$$

Solving the ordinary differential equation (2.6) with boundary conditions (2.7) in the interval $0 \le X \le 1$, we obtain

$$Y^*(X) = A_0(\exp(i(\Phi_0 - \Phi)X) - J(\Phi)N(X, \Phi))P_0(\Phi)$$

$$N(X, \Phi) = (1 - \psi\sigma_2^2)\frac{\operatorname{sh}(\sigma_1(1 - X)) + \exp(i(\Phi_0 - \Phi))\operatorname{sh}(\sigma_1 X)}{2(\sigma_1^2 + \sigma_2^2)(\sigma_1(\cos(\Phi_0 - \Phi) - \operatorname{ch}\sigma_1)} -$$

$$-(1 + \psi\sigma_1^2)\frac{\sin(\sigma_2(1 - X)) + \exp(i(\Phi_0 - \Phi))\sin(\sigma_2 X)}{2(\sigma_1^2 + \sigma_2^2)(\sigma_2(\cos(\Phi_0 - \Phi) - \cos\sigma_2)} \tag{2.8}$$

$$2\sigma_{2,1}^2 = ((\mathrm{B} - \Gamma)^2 + 4\mathrm{A})^{1/2} \pm (\mathrm{B} + \Gamma)$$

$$J(\Phi) = K(\Phi)D(\Phi)/(K(\Phi) + D(\Phi)), \quad D(\Phi) = 1/N(0, \Phi)$$

$$P_n(\Phi) = \frac{1 + \psi((\Phi - \Phi_n)^2 - \mathrm{B})}{(\Phi - \Phi_n)^4 - (\mathrm{B} + \Gamma)(\Phi - \Phi_n)^2 + \mathrm{B}\Gamma - \mathrm{A}}, \quad \Phi_n = \Phi_0 + 2\pi n$$

Using the Fourier expansion

$$\exp(i(\Phi - \Phi_0)X)N(X, \Phi) = \sum_{m=-\infty}^{+\infty} Q_m(\Phi)\exp(i2\pi mX)$$

$$Q_m(\Phi) = \frac{1 + \psi((\Phi - \Phi_m)^2 - \mathrm{B} - \Gamma)}{(\Phi - \Phi_m)^4 - (\mathrm{B} + \Gamma)(\Phi - \Phi_m)^2 + \mathrm{B}\Gamma - \mathrm{A}} \tag{2.9}$$

and then taking inverse transformations, we obtain a dimensionless quantity

$$Y(X, T) = \frac{A_0}{2\pi}\exp(i\Phi_0 X)\int_{-\infty}^{+\infty}\left(1 - J(\Phi)\sum_{m=-\infty}^{+\infty} Q_m(\Phi)\exp(i2\pi nX)\right)\times$$
$$\times P_0(\Phi)\exp(i\Phi(T - X))d\Phi \tag{2.10}$$

The poles of the integrand in (2.10) are investigated in the same way as in [4, 6]. The investigation shows that the integrand has no real poles. The integral (2.10) exists and is unaffected if the quantities T and X are simultaneously increased by one. When that is done the function $\exp(i\Phi_0 X)$ and together with it $Y(X, T)$ are multiplied by $\exp(i\Phi_0)$. Thus, the quantity $Y(X, T)$ satisfies condition (2.2), and so, for any T and X, it is a solution of the problem. At the point of application of the harmonic force the quantity $T - X$ vanishes. In the moving system of coordinates it is a constant, and formula (2.10) represents the vertical displacement of the track as a generalized Fourier series.

3. STEADY VIBRATIONS OF A WHEEL AND TRACK

Suppose a wheel of mass m_0 is moving along a track without detachment, at a constant velocity υ_0. We shall investigate the steady vertical vibrations of the wheel and the track due to a vertical harmonic force $a_0\exp(i\omega_0 t)$. The vertical displacement of the track satisfies condition (2.2). The vertical displacement of the wheel $y_0(t)$ and the wheel–track interaction force $f(t)$ satisfy the conditions

$$y_0(t + l/\upsilon_0) = \exp(i\Phi_0)y_0(t), \quad f(t + l/\upsilon_0) = \exp(i\Phi_0)f(t)$$

according to which each of these quantities is obtained by evaluating the product of $\exp(i\omega_0 t)$ and a periodic function of frequency equal to the sleeper passing frequency υ_0/l. Thus, the wheel–track interaction force $f(t)$ may be represented as a generalized Fourier series

$$f(t) = a_0\sum_{m=-\infty}^{+\infty} F_m\exp\left(i\left(\frac{2\pi m\upsilon_0}{l} + \omega_0\right)t\right)$$

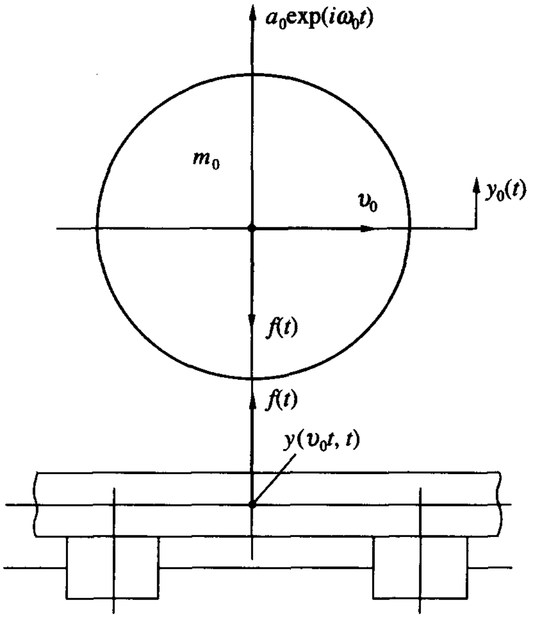

Fig. 3

Throughout, m will denote an integer variable. The dimensionless coefficients F_m of the series have to be determined. The directions of the forces applied to the wheel and the track are shown in Fig. 3. The vertical displacement of the wheel is bounded and is determined by the equation

$$m_0\frac{d^2y_0(t)}{dt^2} = a_0(1-F_0)\exp(\mathrm{i}\omega_0 t) - a_0\sum_{m\neq 0}F_m\exp\left(\mathrm{i}\left(\frac{2\pi m v_0 t}{l}+\omega_0\right)t\right)$$

Integrating the differential equation and changing to the dimensionless vertical displacement of the wheel $Y_0(T) = y_0(t)/l$, mass $M_0 = m_0 v_0^2/(EJ)$ and force $F(T) = f(t)l^2/(EJ)$, we obtain

$$F(T) = A_0\sum_{m=-\infty}^{+\infty}F_m\exp(\mathrm{i}\Phi_m T) \tag{3.1}$$

$$Y_0(T) = \frac{A_0}{M_0}\left(\frac{F_0-1}{\Phi_0^2}+\sum_{m\neq 0}\frac{F_m\exp(\mathrm{i}\Phi_m T)}{\Phi_m^2}\right) \tag{3.2}$$

The steady vertical vibrations of the wheel (3.2) are bounded.

Let $Y_n(X, T)$ denote the dimensionless displacement of the track due to the dimensionless force $\exp(\mathrm{i}\Phi_n T)$. Replace Φ_0 by Φ_n, $P_0(\Phi)$ by $P_n(\Phi)$, and $Y(X, T)$ by $Y_n(X, T)$ in Eqs (2.8) and (2.10). Now, substituting $X = 0$ in Eq. (2.8) and taking the inverse transformation, we obtain the dimensionless vertical displacement of the beam:

$$\begin{aligned} Y_n(0, T) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}J^+(\Phi)P_n(\Phi)\exp(\mathrm{i}\Phi T)d\Phi \\ J^+(\Phi) &= D(\Phi)/(K(\Phi)+D(\Phi)) \end{aligned} \tag{3.3}$$

Using expansion (2.10), we obtain

$$Y_n(X, T) = \sum_{m=-\infty}^{+\infty}W(m, n, T-X)\exp(\mathrm{i}\Phi_m X)$$

$$W(m,n,T-X) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} J(\Phi)Q_m(\Phi)P_n(\Phi)\exp(i\Phi(T-X))d\Phi, \quad m \neq n$$

$$W(n,n,T-X) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (1 - J(\Phi)Q_n(\Phi))P_n(\Phi)\exp(i\Phi(T-X))d\Phi$$

Let $Y(X, T)$ denote the dimensionless vertical displacement of the track due to the dimensionless moving force $F(T)$. Then

$$Y(X,T) = A_0 \sum_{n=-\infty}^{+\infty} F_n Y_n(X,T) = A_0 \sum_{n=-\infty}^{+\infty} F_n \sum_{m=-\infty}^{+\infty} W(m,n,T-X)\exp(i\Phi_m X) \tag{3.4}$$

At $X = T$ the last quantity determines the dimensionless vertical displacement of the track at the point of contact with the wheel, and it is equal to $Y_0(T)$. Equating the coefficients of $\exp(i\Phi_m X)$ in the expansions of $Y_0(T)$ and $Y(T, T)$, we obtain an infinite system of linear equations

$$M_0\Phi_0^2 \sum_{n=-\infty}^{+\infty} F_n W(0,n,0) - F_0 = 1, \quad M_0\Phi_m^2 \sum_{n=-\infty}^{+\infty} F_n W(m,n,0) - F_m = 0, \quad m \neq 0 \tag{3.5}$$

for determining the infinite set of unknowns F_m.

The spring-cushioned part of the carriage is separated from the wheel by an elastic spring. In the first approximation, therefore, its action on the wheel is a static load. Consider the vibrations of the wheel due to this static load. We have $\omega_0 = 0$, $\Phi_0 = 0$, $\Phi_m = 2\pi m$, and the vertical displacements of the wheel and track are real numbers. By the first equation of (3.5), we have $F_0 = 1$. The coefficients F_m and F_{-m} are complex conjugates. The real number $F_m\exp(i2\pi mT) + F_{-m}\exp(-i2\pi mT)$ defines the mth harmonic component of the wheel-track interaction.

The results of computations of the dimensionless amplitude of that harmonic, which is equal to $2|F_m|$, as a function of the velocity of motion υ_0 for a wheel of mass $m_0 = 700$ kg, $m = 1, 2$, and track parameters

$$EJ = 3.57 \times 10^6 \text{ Nm}^2, A = 0.006 \text{ m}^2, k' = 0.34, \rho_0 = 48 \text{ kg/m}, l = 0.8 \text{ m}$$
$$u = 40 \times 10^6 \text{ Nm}^2, \rho_1 = 43.6 \text{ kg/m}, r = 26 \times 10^3 \text{ Ns/m}^2$$

are shown in Fig. 4 (the solid curves). The amplitude of the first harmonic reaches its maximum value, comprising 20% of the static load, at a critical velocity 123.8 km/h. At this velocity the frequency of parametric disturbance of the wheel due to periodic variation of the track stiffness [6] equals the sleeper passing frequency of 43 Hz and is identical with the frequency of the free vibrations of the wheel set on the track. Thus, one has parametric resonance. The maximum value of the amplitude of the second harmonic corresponds to a velocity of 61.9 km/h (half the aforementioned critical velocity) and the same frequency of 43 Hz.

These amplitudes, computed ignoring sheer deformation in the rail, are shown in Fig. 4 by the dashed curves. Owing to the lack of the shear deformation, the track stiffness increases by 5% and accordingly there is an increase in the frequency of free vibrations of the wheel. That is why the apexes of the dashed curves are displaced toward high velocities. The significant decrease in amplitude indicates that parametric perturbation of the wheel is associated with that deformation. It should be mentioned that in previous studies [10–13] of parametric vibrations of the track and wheels of railway carriages, shear deformation in the rail was not taken into consideration. Hence the vibrations turned out to be insignificant.

4. THE RESISTANCE OF A RAILWAY TRACK TO UNIFORM MOTION OF A WHEEL

The resistance of a homogeneous railway track to uniform motion of a wheel at constant loads has been studied in [14, 15] ignoring shear deformation in the rail. The motion of the wheel produces a displacement of the track which is stationary in the moving system of coordinates. The resistance of the subrail bedding retards the deformation of the rail. Therefore, the lowest point of the elastic axis of the

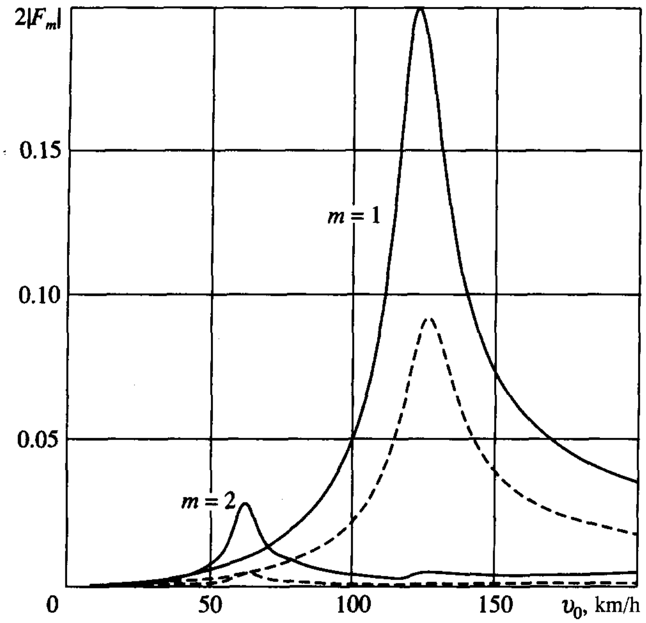

Fig. 4

rail lags behind the moving wheel, which is kept high by a lengthwise force that overcomes the resistance of the track and makes uniform motion of the wheel possible. Thus, the resistance of the track is similar to the wave resistance of water to the motion of a boat.

Shear deformation in the rail causes a break in the elastic axis of the rail at the point of contact with the wheel, so that the approach described above is no longer possible. Below we develop a different approach to computing the resistance of a railway track to the uniform motion of a wheel loaded by a vertical periodic force. The vibrations of the supports are accompanied by the absorption of energy, determined in the form of a quadratic functional and computed using Parseval's equality. The energy is replenished by a lengthwise force which keeps the wheel in uniform motion and is equal to the force of resistance of the track to the motion.

We shall first assume that the vertical displacement of the track has the period $p_2 l$ defined by Eq. (2.3). The average power developed by the vertical force in that period is zero. The motion of the loaded wheel is accompanied by absorption of energy by the sub rail bedding and erosion of ballast. The lengthwise force that keeps the wheel in motion is a single energy source. Equating the power developed by the lengthwise force to the power absorbed by the sub-rail bedding, we determine the lengthwise force and thus also the resistance of the track to the wheel's motion. By (2.3), the pressure of the rail on a sleeper at a point $x_j = jl$ ($j = 0, \pm 1, \pm 2, \ldots$) is equal to $k(j, t) = k(0, t - jl/v_0)$, and $y(x_j, t) = y(0, t - jl/v_0)$.

The instantaneous power developed by the force $k(j, t)$ is computed as the product of that force by the vertical velocity of the rail, $\partial y(x_j, t)/\partial t$. Summing the power over all the sleepers, we determine the instantaneous power

$$N(t) = \sum_{j=-\infty}^{+\infty} k(j, t)\frac{\partial y(x_j, t)}{\partial t}dt$$

expended by the uniformly moving wheel in deforming the track. Averaging the instantaneous power $N(t)$ over the time interval $0 \le t \le p_2 l/v_0$ corresponding to the wheel passing a distance $p_2 l$, we obtain

$$\langle N \rangle = \frac{v_0}{p_2 l}\sum_{j=-\infty}^{+\infty}\int_0^{p_2 l/v_0} k(j, t)\frac{\partial y(x_j, t)}{\partial t}dt$$

Replacing the quantity $t - jp_2/v_0$ in each term of this sum by t and changing the limits of integration, we obtain

$$\langle N\rangle = \frac{\upsilon_0}{p_2 l}\left(\ldots + \int_{-l/\upsilon_0}^{(p_2-1)l/\upsilon_0} + \int_{0}^{p_2 l/\upsilon_0} + \int_{l/\upsilon_0}^{(p_2+1)l/\upsilon_0} + \ldots\right)k(0,t)\frac{\partial y(0,t)}{\partial t}dt$$

The right-hand side of this expression is an infinite sum (in both the positive and negative directions) of integrals whose intervals of integration are of the same length $p_2 l$. The limits of integration of each term are obtained by successively increasing the limits of integration of the first term by l/υ_0. Thus, the result of the integration is equal to the product of p_2 and an integral with two infinite limits. Dividing by p_2, we obtain the equality

$$\langle N\rangle = \frac{\upsilon_0}{l}\int_{-\infty}^{+\infty} k(0,t)\frac{\partial y(0,t)}{\partial t}dt \tag{4.1}$$

which shows that the average power absorbed by all the sleepers in the time l/υ_0 needed by the wheel to transverse the distance between sleepers is equal to the power absorbed by one sleeper as the time varies from $-\infty$ to $+\infty$.

If the vertical displacement of the rail is not periodic, then for sufficiently large p_2 Eq. (2.3) will hold approximately, to within a prescribed accuracy. Equation (4.1) is independent of p_2 and consequently is exact for non-periodic rail displacement as well.

Substituting (2.4) into integral (4.1) and taking into consideration that, at any point x, the vertical displacement $y(x, t)$ of the rail and its derivatives tend to zero as $t \to \pm\infty$, we obtain

$$\langle N\rangle = \frac{\upsilon_0\rho_1}{2}\left(\frac{\partial y(0,t)}{\partial t}\right)^2\Bigg|_{-\infty}^{+\infty} + \upsilon_0 r\int_{-\infty}^{+\infty}\left(\frac{\partial y(0,t)}{\partial t}\right)^2 dt + \frac{\upsilon_0 u y^2(0,t)}{2}\Bigg|_{-\infty}^{+\infty} = \upsilon_0 r\int_{-\infty}^{+\infty}\left(\frac{\partial y(0,t)}{\partial t}\right)^2 dt \tag{4.2}$$

In the linear formulation of the problem considered, there is no lengthwise force keeping the wheel in uniform motion, the vertical displacement of the rail $y(x, t)$ is a small (first-order) quantity, and higher-order small quantities are not taken into consideration. The average power $\langle N\rangle$ defined by the quadratic functional (4.2) is a small (second-order) quantity.

The constant force f_* satisfying the equality $\langle N\rangle = f_*\upsilon_0$ is identical with the average resistance of the track to uniform motion of the wheel. The transforming in the integral (4.2) to dimensionless variables and dividing both sides of the equality by υ_0, we obtain

$$f_* = \upsilon_0 r l\int_{-\infty}^{+\infty}\left(\frac{\partial Y(0,T)}{\partial T}\right)^2 dT \tag{4.3}$$

5. COMPUTATION OF THE RESISTANCE OF THE RAILWAY TRACK

The fact that formulae (4.2) and (4.3) are non-linear complicates the computation of the track resistance to the motion of a wheel loaded by vertical harmonic forces of a different frequency. The computation is simplified by using Parseval's equality [16], which has the following form for the Fourier transformation (2.5)

$$\int_{-\infty}^{+\infty}|Y(X,T)|^2 dT = (2\pi)^{-1}\int_{-\infty}^{+\infty}|Y^*(X)|^2 d\Phi$$

Differentiating the second equality of (2.5) with respect to T and putting $X = 0$, we obtain

$$\frac{\partial Y(0,T)}{\partial T} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} i\Phi Y^*(0)\exp(i\Phi T)d\Phi$$

If the quantities $Y(0, T)$ and $Y^*(0)$ satisfy relation (2.5), then $i\Phi Y^*(0)$ is the Fourier transform of the derivative $\partial Y(0, T)/\partial T$. Applying Parseval's equality to the derivative and substituting the result into Eq. (4.3), we obtain

$$f_* = \frac{\upsilon_0 rl}{2\pi}\int_{-\infty}^{+\infty}\Phi^2|Y^*(0)|^2 d\Phi = \frac{\upsilon_0 rl}{2\pi}\int_{-\infty}^{+\infty}\Phi^2 Y^*(0)\bar{Y}^*(0)d\Phi \tag{5.1}$$

A bar over a symbol denotes the complex conjugate.

Let us compute the average resistance of the railway track to the uniform motion of a wheel loaded by a vertical force

$$a_0\cos(\omega_0 t) = a_0(\exp(i\omega_0 t) + \exp(-i\omega_0 t))/2 \tag{5.2}$$

The terms on the right of Eq. (5.2) are complex conjugates.

Put

$$J^-(\Phi) = D(\Phi)/(K(-\Phi) + D(\Phi))$$

According to Eqs (3.3) and (3.4), the dimensionless displacement of a sleeper, corresponding to the first exponential function on the right of Eq. (5.2), can be written in the form

$$Y(0, T) = A_0\sum_{n=-\infty}^{+\infty}F_n Y_n(0, T) = A_0\sum_{n=-\infty}^{+\infty}\frac{F_n}{2\pi}\int_{-\infty}^{+\infty}J^+(\Phi)P_n(\Phi)\exp(i\Phi T)d\Phi$$

We now take complex conjugates in the last equality and replace Φ by $-\Phi$. This sequence of actions does not change $K(\Phi)$. We now write the dimensionless displacement of the sleeper corresponding to the second term in the form

$$\bar{Y}(0, T) = A_0\sum_{n=-\infty}^{+\infty}\frac{\bar{F}_n}{2\pi}\int_{-\infty}^{+\infty}J^+(-\Phi)P_n(-\Phi)\exp(i\Phi T)d\Phi$$

The dimensionless displacement of the sleeper corresponding to the force (5.2) has the Fourier transformation

$$\frac{A_0}{2}\sum_{n=-\infty}^{+\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty}(F_n J^+(\Phi)P_n(\Phi) + \bar{F}_n J^-(-\Phi)P_n(-\Phi))d\Phi$$

We now take complex conjugates

$$\frac{A_0}{2}\sum_{m=-\infty}^{+\infty}(\bar{F}_m J^-(\Phi)P_m(\Phi) + F_m J^+(-\Phi)P_m(-\Phi))$$

and substitute the product of the last two expressions into formulae (5.1). Changing the order of summation and integration, we can compute the track resistance to the motion of a wheel loaded by the vertical force (5.2) as a quadratic form

$$f_* = \frac{\upsilon_0 rlA_0^2}{4}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}(S_1(m, n)F_mF_n + S_2(m, n)\bar{F}_m\bar{F}_n + S_3(m, n)F_m\bar{F}_n + S_4(m, n)\bar{F}_mF_n)$$

The coefficients of the quadratic form $S_j(m, n)$ $(j = 1, 2, 3, 4)$ are integral, which are computed at the same time as the coefficients $W(m, n, 0)$ of system (3.5).

The results of computations of f_* as a function of the velocity υ_0 of motion of the wheel are shown in Fig. 5. Curve 1 is the track resistance to the motion of a balanced wheel under a static load of 5.78×10^4 N, and curve 2 is the track resistance to the motion of an unbalanced wheel of disbalance 0.5 kg with the same static load. The ratio of the wheel circumference to the sleeper spacing is conventionally taken to be 73/24. The period of vertical vibrations of the wheel corresponds to 24 revolutions of the wheel and to a track length equal to 73 sleeper spacings. In both cases, the track resistance disappears as $\upsilon_0 \to 0$ and increases sharply at a velocity of 123.8 km/h. The first fact is explained

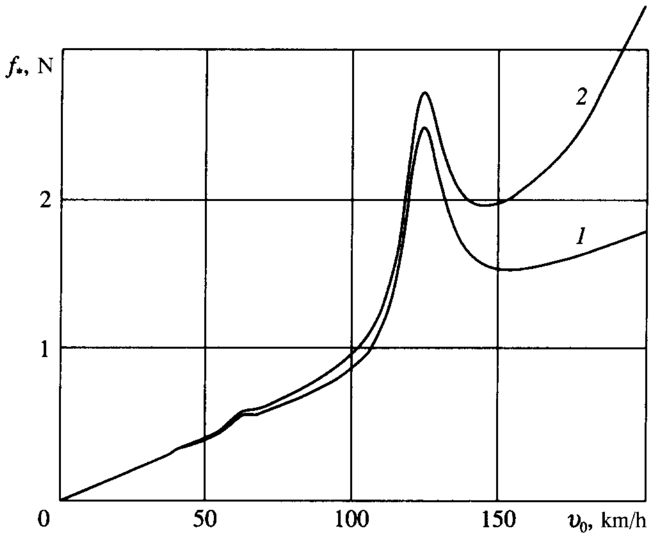

Fig. 5

by the wave nature of the resistance. The second is associated with the parametric resonance of the wheel.

The centrifugal force significantly increases the track resistance to the motion of an unbalanced wheel at velocities exceeding 150 km/h. The critical velocity of the wheel corresponding to the centrifugal force is considerably greater than the latter value.

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